Journal of Engineering Mathematics **46:** 213–226, 2003.

Advanced Kantorovich method for biharmonic problems

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Received 28 May 2002; accepted in revised form 21 February 2003

Abstract. In this paper a method for solving biharmonic problems involving a mixed numerical-analytical approach is described. The algorithm of this method is given and the efficiency of its application for the solution of biharmonic problems is discussed. The recommendations about an application of this method for solving stationary three-dimensional problems in the theory of elasticity are given.

Key words: advanced Kantorovich method, biharmonic problems, deformation of plates

1. Introduction

By the end of the 19th century, the mathematical model for conservative linear three-dimensional problems of static of deformable isotropic bodies was already completely formulated. However, the exact solution for most of the problems (especially applied ones) in a strict statement could be derived only in exceptional cases. This caused the fact that more simple two-dimensional deformation models were widely used. These models allowed to solve actual engineering problems with a reasonable degree of rigor and accuracy. Here we mention two such problems: 1 – bending of a thin isotropic plate under normal loading, 2 – plane-strain state of a thin plate.

The solution of these problems was reduced to solving a biharmonic problem with different boundary conditions and different types of loading. The construction of the adequate solution of the problem depended equally upon the engineering intuition of the researchers and on the level of their mathematical knowledge and ingenuity. In the 20th century, especially during the first half of it, a large group of outstanding 'mathematical-engineers' (or 'engineering-mathematicians') were active. However, here we limit ourselves to refer to such names (in alphabetical order) as Filon, Galerkin, Mitchell and Timoshenko who, in our view, have forever entered into the history of engineering.

In the 20th century the approach to solving biharmonic problem was changed from numerical-analytical tabulating the set of solutions or the physical simulation of the simplest constructive elements to computational research of the strained state of constructions using FEM or BEM. It can be stated that at present the numerical implementation of almost any consistent computing algorithm is not so much a scientific as an economic problem, because it depends on the accessibility to the necessary computer resources. Therefore, the following question arises: why should we in the 21st century pay attention to an 'elementary' biharmonic problem?

We consider that this problem is a fundamental component of that area of knowledge that belongs to classical mathematical physics, mechanics of deformable bodies and engineering mathematics. Analytical and numerical solutions of biharmonic problem obtained in the 20th century will still, for a long time, have significance as a reliable basis for testing when developing new effective methods for the numerical solution of multi-dimensional boundary-value problems.

In this paper we discuss the solutions of a number of biharmonic problems. They were selected to illustrate the features and efficiency of the method which we propose for the solution of linear multi-dimensional problems in the mechanics of a deformable body. In this paper the method is called 'Advanced Kantorovich's Method'. This method is the development of the known Kantorovich's Method of reducing of the $2D$ variational problems to ordinary differential equations.

The paper is organized as follows: Section 2 presents the structure and the algorithm of the proposed numerical-analytical method for solving a biharmonic problem. Section 3 discusses the results of the solutions of some biharmonic problems obtained by using the advanced Kantorovich method. Section 4 presents some general conclusions.

2. Foundation of the Advanced Kantorovich Method

It is possible to define the Advanced Kantorovich Method (AKM) as well as Kantorovich's method of reduction to ordinary differential equations as the class of approximate methods for solving boundary-value problems.The origin of these methods may be associated with the names of Galerkin, Bubnov, Ritz and Trefftz.

The two following steps must be executed when constructing a solution by these methods: first, approximation of the required function by a final series of trial functions for a system chosen beforehand and, secondly, solving of 'the approximating problem'[1] in the assumption, that it can be solved without any problem with a required accuracy.

Let us consider how these stages are realized in AKM by an example of a solution of a biharmonic problem:

$$Lu \equiv \Delta\Delta u = f, \tag{1}$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \tag{2}$$

in the domain $\Omega = \{|x| \leq a, |y| \leq b\}$ (n is normal to the boundary of domain).

In a variational statement this problem is equivalent to the determination of a minimum of the following functional:

$$J(u) = \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}\left[(\Delta u)^2 - 2fu\right]\mathrm{d}x\mathrm{d}y. \tag{3}$$

The required function $u(x, y)$ in the first stage of AKM is presented by a linear combination of functions with separated variables:

$$u(x, y) \cong F_M(x, y) = \sum_{i=1}^{M} X_i(x) Y_i(y), \tag{4}$$

[1]The term is taken from [1, p. 189].

where all functions $X_i(x)$, $Y_i(y)$, $(i = \overline{1, M})$ are unknown.

Such a representation of the required function can be regarded as a generalization of the approximating forms, which are characteristic ones for Galerkin-Ritz methods and Kantorovich's method [2–4]. So, in the Galerkin-Ritz methods the function $u(x, y)$ is approximated as follows:

$$u(x, y) \cong F_{MN}(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} \varphi_i(x) \psi_j(y), \tag{5}$$

where α_{ij} are unknown numerical coefficients and $\varphi_i(x)$, $\psi_j(y)$ are the trial functions, which are usually part of some complete set. It is obvious that, if the eigenfunctions of the problem (1), (2) are chosen as the trial functions, then representation (5) corresponds to a method of expansion about the eigenfunctions.

Developing Galerkin's ideas, Kantorovich has altered representation (5). Here, instead of numerical coefficients α_{ij}, the functional coefficients $X_i(x)$ were entered as unknowns, *i.e.*, the form (5) was transformed to the following expression:

$$u(x, y) \cong F_M = \sum_{i=1}^{M} X_i(x) \psi_i(y). \tag{6}$$

In [5, Section 4.3], by an example concerning Poisson's equation, it is shown that approximation (6) leads to greater solution accuracy than with approximation (5).

The introduction in representation (6) of functions of the second variable $Y_i(y)$ as unknowns generalizes Kantorovich's approach and transforms it to the approximating AKM form (4).

The second stage of solving by AKM is connected with a construction of the 'approximating problem' about the introduced unknown functions $X_i(x)$, $Y_i(y)$, $(i = \overline{1, M})$. For this purpose a variational statement of problem (3) is used. The substitution of the approximation form (4) in the functional (3) leads to the following expression:

$$J_M(F_M) = \frac{1}{2} \int_{-a}^{a} \int_{-b}^{b} \left[\left(\sum_{i=1}^{M} X_i'' Y_i \right)^2 + 2 \left(\sum_{i=1}^{M} X_i' Y_i' \right)^2 + \left(\sum_{i=1}^{M} X_i Y_i'' \right)^2 \right] \mathrm{d}x \mathrm{d}y$$
$$- \int_{-a}^{a} \int_{-b}^{b} f \left(\sum_{i=1}^{M} X_i Y_i \right) \mathrm{d}x \mathrm{d}y \,. \tag{7}$$

The condition of an extremum of this functional has the form:

$$\delta J_M(X_1, X_2, ..., X_M, Y_1, Y_2, ..., Y_M) = 0\,,$$

i.e.,

$$\delta J_M = \sum_{k=1}^{M} \delta_{X_k} J_M \delta X_k + \sum_{k=1}^{M} \delta_{Y_k} J_M \delta Y_k = 0\,, \tag{8}$$

where $\delta_{X_k} J_M$ and $\delta_{Y_k} J_M$ are the partial variations of the functional J_M by the functions, which are specified in a lower index. The independence of all functions $X_k(x)$ and $Y_k(y)$ in (8) allows us to write them as follows:

$$\delta_{X_k} J_M = 0\,, \quad \delta_{Y_k} J_M = 0\,, \ (k = \overline{1, M}). \tag{9}$$

Application of the well-known technique of the calculus of variations to each of the equations in (9) by accounting for the boundary conditions (2) generates a set of equations in connection with the functions $X_i(x)$ and $Y_i(y)$ ($i = \overline{1, M}$)

$$\sum_{i=1}^{M} \left(X_i^{IV} \int_{-b}^{b} Y_i Y_k \mathrm{d}y - 2X_i^{''} \int_{-b}^{b} Y_i' Y_k' \mathrm{d}y + X_i \int_{-b}^{b} Y_i^{''} Y_k^{''} \mathrm{d}y \right) = \int_{-b}^{b} f Y_k \mathrm{d}y, \quad |x| < a,$$

$$X_k\,(\pm a) = 0, \quad X_k'\,(\pm a) = 0, \tag{10}$$

$$\sum_{i=1}^{M} \left(Y_i^{IV} \int_{-a}^{a} X_i X_k \mathrm{d}x - 2Y_i^{''} \int_{-a}^{a} X_i' X_k' \mathrm{d}x + Y_i \int_{-a}^{a} X_i^{''} X_k^{''} \mathrm{d}x \right) = \int_{-a}^{a} f X_k \mathrm{d}x, \quad |y| < b,$$

$$Y_k\,(\pm b) = 0, \quad Y_k'\,(\pm b) = 0, \quad \left(k = \overline{1, M}\right). \tag{11}$$

This system consists of two one-dimensional boundary-value problems about the variable x (the problem (10) and about the variable y (the problem (11)).

Problem (10) is formulated in terms of unknown functions $X_i(x)$, ($i = \overline{1, M}$). Its coefficients and the free terms contain the functions $Y_i(y)$ in the form of definite integrals. Problem (11), on the contrary, is formulated in terms of the functions $Y_i(y)$, ($i = \overline{1, M}$). Its coefficients and the free terms contain the functions $X_i(x)$ as definite integrals. Thus, the interrelation of the one-dimensional problems (10) and (11) is expressed by its coefficients and free terms.

Let us notice the differences of a 'approximating problem' of AKM in a comparison with 'approximating problems' in Galerkin-Ritz's and Kantorovich's methods. So, the 'approximating problem' in the Galerkin-Ritz methods is a system of algebraic equations about unknown numerical coefficients a_{ij} from (5). In Kantorovich's method this problem is a system of ordinary differential equations in terms of functions $X_i(x)$ from (6), *i.e.*, one one-dimensional problem. This problem is identical to the problem (10), if the unknown functions $Y_i(y)$ in (10) are replaced by the functions $\psi_i(y)$ from (6). Thus, the 'approximating problem' of Kantorovich's method is a special case of the 'approximating problem' of AKM.

The way to construct an AKM 'approximating problem' to obtain a differential statement of $3D$ stationary boundary-value problems in the theory of elasticity is described [6, 7].

For the determination of the functions $X_i(x)$, $Y_i(y)$ ($i = \overline{1, M}$) from a system of one-dimensional problems (10), (11) in AKM, the following iterative process is used, in which n is a step of iterations:

$$\sum_{i=1}^{M} \left(A_{4ik}^{n-1} \left(X_i^n\right)^{IV} - 2A_{2ik}^{n-1} \left(X_i^n\right)^{''} + A_{0ik}^{n-1} X_i^n \right) = A_k^{n-1} \quad |x| < a, \tag{12}$$

$$X_k^n = 0, \quad \left(X_k^n\right)' = 0; \quad \left(k = \overline{1, M}\right) \mathrm{at}\, x = \pm a;$$

$$\sum_{i=1}^{M}\left(B_{4ik}^{n}\left(Y_i^n\right)^{IV}-2B_{2ik}^{n}\left(Y_i^n\right)''+B_{0ik}^{n}Y_i^n\right)=B_k^n \quad |y|<b, \tag{13}$$

$$Y_k^n=0, \quad \left(Y_k^n\right)'=0; \quad \left(k=\overline{1,M}\right) at y=\pm b; \quad (n=1,2,....)\,.$$

Here

$$A_{4ik}^{n-1}=\int_{-b}^{b}Y_i^{n-1}Y_k^{n-1}\mathrm{d}y, \quad A_{2ik}^{n-1}=\int_{-b}^{b}\left(Y_i^{n-1}\right)'\left(Y_k^{n-1}\right)'\mathrm{d}y,$$

$$A_{0ik}^{n-1}=\int_{-b}^{b}\left(Y_i^{n-1}\right)''\left(Y_k^{n-1}\right)''\mathrm{d}y, \quad A_k^{n-1}=\int_{-b}^{b}fY_k^{n-1}\mathrm{d}y, \tag{14}$$

$$B_{4ik}^{n}=\int_{-a}^{a}X_i^nX_k^n\mathrm{d}x, \quad B_{2ik}^{n}=\int_{-a}^{a}\left(X_i^n\right)'\left(X_k^n\right)'\mathrm{d}x,$$

$$B_{0ik}^{n}=\int_{-a}^{a}\left(X_i^n\right)''\left(X_k^n\right)''\mathrm{d}x, \quad B_k^n=\int_{-a}^{a}fX_k^n\mathrm{d}x. \tag{15}$$

It is assumed that in each iteration step the solution of any separate one-dimensional problem can be found (analytically or numerically) with sufficient accuracy.

As an initial form of the functions $Y_k^0(y)$, $(k=\overline{1,M})$, can be chosen any linearly independent functions and it is not required to satisfy the boundary conditions at $y=\pm b$.

Remark. The authors have no complete proof of convergence of the explained iterative process. A heuristic confirmation of its legitimacy can be obtained by experience of the solution of some $2D$ and $3D$ of stationary problems of mathematical physics [6–9].

The final determination of the function $u\,(x,y)$ of the initial problem (1), (2) is constructed as a practical limit of a sequence $F_M(x,y)$ by increasing the number of terms $M=1,2,...$ in expression (4).

Retaining one term of a series in representation (4) makes it possible to obtain the approximate problem solution in analytical form. Let us give this solution to make the exposition of the AKM structure more clear.

Assuming that the required solution is represented in the form (4) for $M=1$, we assume the following approximation for the function $u(x,y)$:

$$u\,(x,y)\cong X\,(x)\,Y\,(y)\;, \tag{16}$$

where the two functions $X(x)$ and $Y(y)$ are unknown.

These functions are determined by solving the next system of the two one-dimensional problems:

$$X^{IV}-2A_xX''+B_xX=C_x, \quad |x|<a, \tag{17}$$

$$X\,(\pm a)=0,\;\; X'\,(\pm a)=0\,; \tag{18}$$

$$Y^{IV} - 2A_y Y'' + B_y Y = C_y, \qquad |y| < b, \tag{19}$$

$$Y(\pm b) = 0,\ \ Y''(\pm b) = 0\,, \tag{20}$$

where

$$A_x = \frac{\int_{-b}^{b} \left(Y''\right)^2 \mathrm{d}y}{\int_{-b}^{b} Y^2 \mathrm{d}y}, \quad B_x = \frac{\int_{-b}^{b} \left(Y''\right)^2 \mathrm{d}y}{\int_{-b}^{b} Y^2 \mathrm{d}y}, \quad C_x = \frac{\int_{-b}^{b} fY \mathrm{d}y}{\int_{-b}^{b} Y^2 \mathrm{d}y}\,, \tag{21}$$

$$A_y = \frac{\int_{-a}^{a} \left(X''\right)^2 \mathrm{d}x}{\int_{-a}^{a} X^2 \mathrm{d}x}, \quad B_y = \frac{\int_{-a}^{a} \left(X''\right)^2 \mathrm{d}x}{\int_{-a}^{a} X^2 \mathrm{d}x}, \quad C_y = \frac{\int_{-a}^{a} fX \mathrm{d}x}{\int_{-a}^{a} X^2 \mathrm{d}x}\,. \tag{22}$$

The common solution of Equations (17), (19) has the usual form:

$$X(x) = D_{1x}\varphi_{1x}(x) + D_{2x}\varphi_{2x}(x) + D_{0x}\,, \tag{23}$$

$$Y(y) = D_{1y}\varphi_{1y}(y) + D_{2y}\varphi_{2y}(y) + D_{0y}\,, \tag{24}$$

The functions $\varphi_{1x}(x)$, $\varphi_{2x}(x)$, $\varphi_{1y}(y)$, $\varphi_{2y}(y)$ are determined in accordance with the boundary conditions (18), (20) and taking into account the symmetry of the problem about the coordinate axis by the following expressions:

$$\varphi_{1x}(x) = \sinh(\alpha_x x)\sin(\beta_x x)\,, \qquad \varphi_{2x}(x) = \cosh(\alpha_x x)\cos(\beta_x x)\,,$$

$$\alpha_x = \sqrt{B_x}\cos\left(\frac{\phi_x}{2}\right), \qquad \beta_x = \sqrt{B_x}\sin\left(\frac{\phi_x}{2}\right), \quad \phi_x = \arctan\left(\frac{\sqrt{B_x - A_x^2}}{A_x}\right), \tag{25}$$

$$\varphi_{1y}(y) = \sinh\left(\alpha_y y\right)\sin\left(\beta_y y\right), \qquad \varphi_{2y}(y) = \cosh\left(\alpha_y y\right)\cos\left(\beta_y y\right),$$

$$\alpha_y = \sqrt{B_y}\cos\left(\frac{\phi_y}{2}\right), \qquad \beta_y = \sqrt{B_y}\sin\left(\frac{\phi_y}{2}\right), \quad \phi_y = \arctan\left(\frac{\sqrt{B_y - A_y^2}}{A_y}\right). \tag{26}$$

The coefficients D_{ix} and D_{iy} $(i = 0, 1, 2)$ are determined by the formulae

$$D_{0x} = \frac{C_x}{B_x}\,, \qquad D_{1x} = \frac{D_{0x}\varphi'_{2x}(a)}{\varphi_{2x}(a)\,\varphi'_{1x}(a) - \varphi_{1x}(a)\,\varphi'_{2x}(a)}\,,$$

$$D_{2x} = \frac{-D_{0x}\varphi'_{1x}(a)}{\varphi_{2x}(a)\,\varphi'_{1x}(a) - \varphi_{1x}(a)\,\varphi'_{2x}(a)}\,, \tag{27}$$

$$D_{0y} = \frac{C_y}{B_y}\,, \qquad D_{1y} = \frac{D_{0y}\varphi'_{2y}(b)}{\varphi_{2y}(b)\,\varphi'_{y}(b) - \varphi_{1y}(b)\,\varphi'_{2y}(b)}\,,$$

$$D_{2y} = \frac{-D_{0y}\varphi'_{1y}(b)}{\varphi_{2y}(b)\,\varphi'_{y}(b) - \varphi_{1y}(b)\,\varphi'_{2y}(b)}\,. \tag{28}$$

Table 1. On the convergence of the iterative process (analytical solution, monomial approximation).

A_x^0; B_x^0; C_x^0	n	$X_n(0) \times 10^3$	$Y_n(0) \times 10^3$	$u_n(0,0) \times 10^3$
2;100;10	1	1·999	6·466	1·292
	2	1·954	6·465	1·264
				1·263
	3	1·954		1·263

Table 2. Deflection and bending moment for a uniform loaded clamped plate.

M	$w(0,0) \times 10^3 \frac{D}{qa^4}$	$-M_x(0,0) \times 10^2 \frac{1}{qa^2}$	$-M_x(\frac{a}{2},0) \times 10^2 \frac{1}{qa^2}$
1	1·263	2·263	5·223
2	1·265	2·292	5·150
3	1·265	2·295	5·130
[15]	1·26	2·31	5·13

To determine the functions $X(x)$ and $Y(y)$, the following iterative procedure is used. In the beginning some initial values of the coefficients A_x^0, B_x^0, C_x^0, satisfying the condition $B_x^0 - (A_x^0)^2 > 0$, are chosen. Then by (23), with account taken of (25) and (27), the function $X^0(x)$ is calculated. Further, by the function $X^0(x)$ and its derivative we determine the coefficients A_y^0, B_y^0, C_y^0 from (22) and the function $Y^0(y)$ from (24), (26), (28). The function $Y^0(y)$ and its derivative are used for the calculation of the coefficients A_x^1, B_x^1, C_x^1 from (21) and so on until the process will converge.

Remark. In 1947 in an investigation of rectangular-plate bending, Vlasov has suggested the idea to use the iterates for the refinement of the monomial approximation (16) ([10]). Later this idea was realized in [11–14].

The described iterative process converges sufficiently rapidly, independently of the choice of the initial approximation. As an example, we give the results of the problem (1), (2) for the square $a/b = 1$, at $f = 1$ in each step of the iteration. The maximal values of the functions $X^n(x)$, $Y^n(y)$, $u^n(x, y)$, which are calculated according to the formulas (23), (24), (16), are given in Table 1.

As for the problem on the plate bending, the obtained solution $u(x, y)$ is the deflection $w(x, y)$ of the clamped square plate with side a under a uniform load q. These results are compared with data given in [15, Section 6.44]. A comparison of results is given in Table 2 for the deflection at the centre of the plate and for the bending moment M_x at the centre and on the middle of the clamped side, which were obtained by Evans and AKM ($D = Eh^3/12\left(1 - \nu^2\right)$; E is Young's modulus, ν is the Poisson's ratio, h is the plate thickness, $\nu = 0{\cdot}3$). The AKM solutions are given when a different number of the terms M in approximation series (4) are retained.

As can be seen from Table 2, the monomial approximation of the AKM coincides with Evans's calculations for the maximum value of the deflection, and the error in the moment value is about 2%. When three terms of the series (4) of the AKM are retained, the differences in the moments do not exceed 0·5%.

Remark. In the Advanced Kantorovich Method for multi-dimensional boundary-value problems for a number of independent variables $N > 2$, the required solution $u(x_1, x_2, \ldots, x_N)$ is represented in the form:

$$u\,(x_1, x_2, ..., x_N) \cong F_M = \sum_{i=1}^{M} X_{1i}\,(x_1)\,X_{2i}\,(x_2)\cdots X_{Ni}\,(x_N),$$

where all functions $X_{1i}(x_1)$, $X_{2i}(x_2), \ldots, X_{Ni}(x_N)$ are unknown.

In this case the ‘approximating problem’ is a system of N one-dimensional problems about each of the variables x_n $(n = \overline{1, N})$.

3. Solution of a test problem

In this section we give the results of some biharmonic problems solved by the Advanced Kantorovich Method (AKM). The problems are chosen to illustrate the efficiency of AKM in the class of mechanical problems under consideration.

3.1. Bending of a simple supported rectangular plate

The classic solution of the problem for bending of a rectangular plate ($\Omega = \{0 \le x \le a;\ 0 \le y \le b\}$) under arbitrary normal load action $q = q(x, y)$ has the following form (Navier solution, 1820):

$$w(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \sin\alpha_i x \sin\beta_j y\,, \tag{29}$$

where $\alpha_i = \frac{i\pi}{a}$, $\beta_j = \frac{j\pi}{b}$, $a_{ij} = \frac{a'_{ij}}{D\left(\alpha_i^2+\beta_j^2\right)^2}$, and a'_{ij}- are the coefficients of the load function expansion in a double trigonometrical Fourier series.

The result (29) is typical for boundary-value problem solved by the classical method of expansion about eigenfunctions (Fourier-series expansion). Here this solution is used as a basis for efficiency estimation of the AKM and Kantorovich’s method in solving problem of plate bending under the loads of different kind.

Application of Kantorovich’s method was based on using an approximation of the required function $w(x, y)$ by the following form:

$$w\,(x, y) = \sum_{j=1}^{\infty} X_j\,(x) \sin\beta_j y \tag{30}$$

(It is worth noting that M. Levy has used precisely this form for the problem on the bending of a plate with two simply supported opposite sides as long ago as 1899).

In the first variant of the problem we considered a square plate with side a under the action of a suitably chosen normal load which yields a sharp localized deflection of the following form:

$$w^*\,(x, y) = \sin^n \alpha x \sin^m \beta y; \quad (n = m = 20,\ \alpha = \beta = \frac{\pi}{a})\,. \tag{31}$$

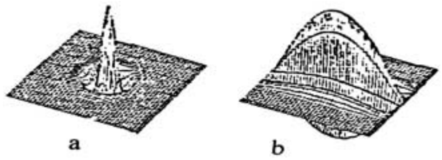

Figure 1. Loading forms corresponding to solution (31): (a) $n = m = 20$; (b) $n = 20, m = 1$.

Table 3. Dependence of the deflection, bending moment and errors at the point $P(0{\cdot}3a, 0{\cdot}3a)$ on the number of the terms of the approximation series in the Fourier method, Kantorovich's method and AKM.

Number terms	$w(P) \times 10^4 \frac{D}{a^4}$	ε_w, %	$-M_x(P) \times 10^5 \frac{1}{a^2}$	ε_M, %
Fourier-series expansion				
1×1	$7{\cdot}7500 \times 10^2$	$3{\cdot}6 \times 10^3$	$-9{\cdot}943$	$3{\cdot}1 \times 10^2$
5×5	$9{\cdot}0545 \times 10^2$	$9{\cdot}6 \times 10$	1·4733	$6{\cdot}9 \times 10$
7×7	2·3682	$1{\cdot}4 \times 10$	4·7135	2·3
8×8	2·0901	0·43	4·8170	0·11
9×9	2·0788	0·11	4·8248	0·056
10×9	2·0800	0·058	4·8214	0·014
Exact solution	2·0812		4·8221	
Kantorovich's method				
1	$4{\cdot}0161 \times 10$	$1{\cdot}8 \times 10^3$	$7{\cdot}1461 \times 10$	$1{\cdot}4 \times 10^3$
3	$-3{\cdot}9658 \times 10^{-1}$	$1{\cdot}2 \times 10^2$	1·5752	$6{\cdot}7 \times 10$
5	4·3425	$7{\cdot}9 \times 10$	2·4037	$5{\cdot}0 \times 10$
7	2·2200	6·68	4·9765	3·2
8	2·0856	0·21	4·8265	0·09
9	2·0800	0·058	4·8213	0·016
Exact solution	2·0812		4·8221	
Advanced Kantorovich method				
1	2·0803	$4{\cdot}8 \times 10^{-3}$	4·8209	$2{\cdot}1 \times 10^{-3}$
Exact solution	2·0812		4·8221	

The form of the load is shown in Figure 1a.

In Table 3 lists the values of the deflection $w(x, y)$ and bending moment $M_x(x, y)$ at the point P with coordinate $(0{\cdot}3a, 0{\cdot}3a)$. The calculations were carried out by method of expansion, Kantorovich's method and AKM for a various number of terms in the approximation forms. We also give the errors of every method with respect to the values of $w^*(P)$ and $M_x^*(P)$ which were calculated in accordance with (31) at $x = 0{\cdot}3a$, $y = 0{\cdot}3a$. The relative value of the deflection $\frac{w^*(P)}{w^*(0{\cdot}5a, 0{\cdot}5a)} \approx 0{\cdot}0002$ at this point and for this reason the error estimate is sufficiently obvious.

It is clear that in each case the AKM is more efficient than Kantorovich's method, which is in its turn performs better than the method of expansion about eigenfunctions. Actually, to

Table 4. Maximum values of the deflection, bending moment and errors at monomial approximation in the Fourier method, Kantorovich's method and AKM.

Method	$w_{\max}\frac{D}{a^4}$	ε_w, %	$-M_{\text{xmax}} \times 10^2 \frac{1}{a^2}$	ε_M, %
Fourier-series	$1{\cdot}184 \times 10^{-1}$	$8{\cdot}44 \times 10^3$	$1{\cdot}519 \times 10^{-2}$	$9{\cdot}94 \times 10$
Kantorovich method	$3{\cdot}441 \times 10^{-1}$	$6{\cdot}66 \times 10^2$	$6{\cdot}894 \times 10^{-1}$	$7{\cdot}31 \times 10$
AKM	1·000	0·0	2·566	0·0
exact solution	1·000		2·566	

calculate the deflection to within an error $\varepsilon \approx 0{\cdot}05\%$ in accordance with method of expansion, we considered 10×9 terms of the Fourier series, in Kantorovich's method we took into account 9 terms in the form (30), and when employing AKM it was sufficient to take only one term in the approximation (4) and three iteration steps for solving the 'approximate problem' to carry out.

Next in Table 4 the maximum values of the deflection and bending moment obtained by these methods at monomial approximation in (4), (29), (30) are given.

The form of the load for the second variant is shown in Figure 1b. It was found that for the calculation of the deflection with an error up to ε by the expansion method one should use 9×1 terms of the Fourier series and in Kantorivich's method and AKM it is sufficient to take only one term of the series in the approximation forms (30) and (4).

In the third variant of the problem the normal load corresponds to a deflection of the following form: $w^*(x, y) = \sin \alpha x \sin \alpha y$. It is obvious that for solving the problem it sufficed to consider only one term in the approximate form of each method.

3.2. Bending of a Clamped Rectangular Plate

As in [16], in this paper we deal with the problem the bending of a plate that is clamped along its contour with the ratio of its sides being defined by $a = 2b$ under a load $q(x, y) = q_0 =$ const. This makes possible for us to compare the results of solving this problem by AKM with the analytical solution obtained by the superposition method [16], as a standard for comparison.

We compare the distribution functions of the normal reactions V_x and V_y at the clamped contour of the plate and its corner points:

$$V_x(y) = D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=a}, \qquad V_y(x) = D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=b}.$$

Such comparison is a strict test for the offered method. Indeed, since the first derivative of the function w is discontinuous in the corner points, an inarbitrary numerical method leads to a sufficiently accurate solution near a corner.

In Table 5 we give the values of the normal reactions in the middle of the clamped sides $x = a$, $y = b$ and in corner points (columns 2–5); the values of total normal reactions on the sides $x = a$, $y = b$ are given in columns 6, 7; the value of the sum reactions to verify the conditions of the static equilibrium of the plate is in column 8 and the values of the solution error relative to the condition of the static equilibrium in column 9. The data in the rows (1–3) are calculated for a various number of terms M in the representation (4). In the rows 4 and

Table 5. Normal reactions at the clamped sides of the rectangular plate, $a = 2b$, according to method of superposition.

M	$\frac{V_x(0)}{q_0\, b}$	$\frac{V_x(b)}{q_0\, b}$	$\frac{V_y(0)}{q_0\, b}$	$\frac{V_y(a)}{q_0\, b}$	$\frac{R_x}{q_0\, ab}$	$\frac{R_y}{q_0\, ab}$	$\frac{R_x+R_y}{p_0\, ab}$	ϵ, %
1	0·965	0	1·137	0	0·519	1·467	1·986	0·7
2	0·948	0	1·031	0	0·510	1·494	2·004	0·2
3	0·932	0	1·032	0	0·500	1·502	2·002	0·1
[16] (1)	0·931	−0·035	1·037	−0·036	0·499	1·501	2·000	0·0
[16] (2)	0·928	0	1·032	0	0·502	1·498	2·000	0·0

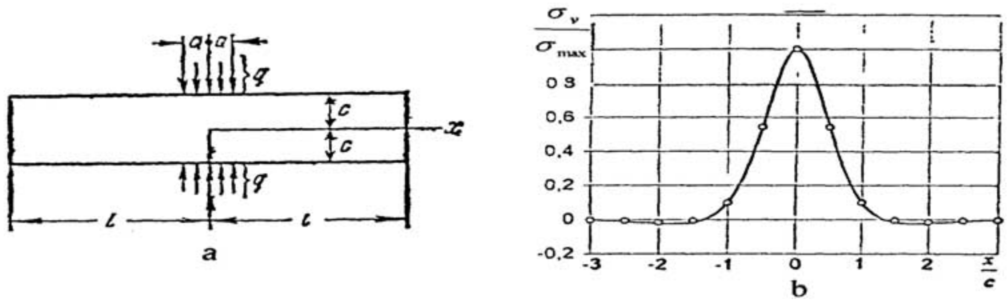

Figure 2. Distribution of the normal stresses in a beam with a localized load: (a) a schematic sketch of the problem; (b) distribution of the stresses $\frac{\sigma_y}{\sigma_{\max}}$ (solid line corresponds to (32), dotted one is AKM).

5 of this table we give the analytical results in accordance with the superposition method for two variants of its realization [16].

A comparison of the results obtained by AKM with the data from [16] permit to draw the following conclusions.

Calculations in accordance with AKM at $M = 1$ yields an error with the respect to the condition of the static equilibrium that is approximately equal to 0·7 %; however, the functions of the normal reactions V_x and V_y on the clamped sides, as distinguished from [16], are monotonous. Calculations for $M = 3$ by AKM leads to a decrease of the mentioned error to 0·1%. At the same time the normal reactions on the contour are non-monotonic functions with negative values close to the corner points that correspond to the results [16].

3.3. Plane stressed state of a beam with a rectangular cross-section

The different variants of this problem were considered by many scientists in the first half of the last century. Here we report the solution in accordance with AKM of the problem which was taken from Filon's paper [17].

We examined a long narrow strip under a load of intensity q, which is localized on the short section of the boundary contour, the length of which is $2a$, so that $P = 2aq$ (Figure 2).

According to [17] the distribution of the normal stresses in the plane $y = 0$ is determined by the following formula:

$$\sigma_y = -\frac{qa}{l} - \frac{4a}{\pi}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi a}{l}\,\frac{m\pi c}{l}\cosh\frac{m\pi c}{l} + \sinh\frac{m\pi c}{l}}{\sinh\frac{2m\pi c}{l} + 2\frac{m\pi c}{l}}\cos\frac{2m\pi x}{l}. \tag{32}$$

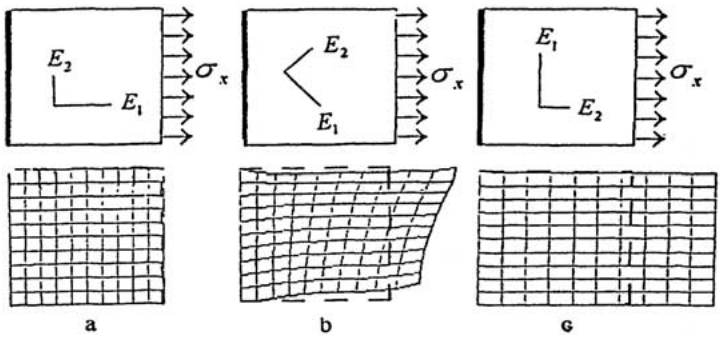

Figure 3. The form of the deformed anisotropic plate: (a) $\psi = 0$; (b) $\psi = 0{\cdot}75\pi$; (c) $\psi = 0{\cdot}.5\pi$ (the dashed line represents the contour of the non-deformed plate).

Table 6. Values of the relative stress $\frac{\sigma_y}{\sigma_{\max}}$ at $\frac{x}{c} = 0{\cdot}5$ and $\frac{x}{c} = 1{\cdot}0$ and the number of the approximation terms according to (32) and AKM.

$\frac{x}{c}$	Stress $\frac{\sigma_y}{\sigma_{\max}}$		Number terms	
	(32)	AKM	(32)	AKM
0·5	0·543	0·543	14	3
1·0	0·099	0·098	10	3

By using AKM we formulate the biharmonic equation (1) about the stress function $\phi(x, y)$ in the domain $\Omega = \{|x| \leq l,\ |y| \leq c\}$. The boundary conditions have the following form: the sides $y = \pm c$ are loaded by a localized stress σ_y of the given aspect and the sides $|x| = l$ are free from load. The calculations were carried out for the data: $c = 40$, $l = 200$, $a = 1{\cdot}2$.

The distribution of the normal stresses $\sigma_y/\sigma_{\max}$ according to AKM and to [17] is shown in Figure 2b.

In Table 6 we give the number of terms of the series (32) and in approximate AKM form, which are needed for the calculation of the function σ_y with a relative error $\varepsilon < 1{\cdot}1\%$.

It is immediately obvious that using AKM for the solution of this problem is preferred to the method proposed in [17].

3.4. Plane deformation of an anisotropic plate

The three foregoing problems had known solutions which were used in discussing the efficiency of AKM. In certain sense the problem that will be discussed now is new, although the qualitative results of its solution are obvious.

The problem on plane deformation of a square plate with side a is examined. One side ($x = 0$) of its boundary contour is clamped, but the opposite side ($x = a$) is under the action of the uniform load σ_x. The sides $y = 0$, $y = a$ of the contour are free. The plate is made of orthotropic material and its main direction of elasticity E_1 and E_2 can be oriented arbitrarily about the boundary contour. In this case the latter leads to the passage from classical biharmonic equation to its generalized form [18, p. 136].

The material of the plate is carbon-fibre-reinforced plastic with the following properties: $\frac{E_1}{E_2} = 40$, $\frac{G_{12}}{E_2} = 0{\cdot}5$, $\nu = 0{\cdot}25$.

The orientation of the material about the plate boundary is defined by the angle ψ between the direction of the maximum modulus of elasticity E_1 and the positive direction of the coordinate axis $0x$. Figure 3 illustrates the form of the deformed plate for the various values of ψ.

It is clear that minimum displacements happen when the directions of the load σ_x and maximum modulus E_1 coincide, maximum displacements correspond to the orientation $\psi = 0{\cdot}5\pi$ and appreciable distortion of the plate occurs at $\psi = 0{\cdot}75\pi$.

4. Conclusions

Completing the presentation of the Advanced Kantorovich Method, it is safe to say that this method is an efficient means for solving many-dimensional stationary problems of the theory of elasticity. The advantages of AKM have been shown for examples of classical problems such as the bending of a thin rectangular plate and the plane strain problem. Let us mention these advantages:

1. The problem of adequate choice of trial functions is absent.
2. AKM allows to reveal fine mechanical effects. For this it is sufficient to take three terms in the approximative form (4) and to execute four steps of the iterative procedure.
3. The dimensions of all internal algebraic and ordinary differential equations in the AKM algorithm grows as linear function with respect the number of independent variables in the region.

In other approximate approaches (for instance Galerkin-Ritz's methods, the eigenfunctions expansion) the specified dependence is exponential. This feature allows one to use AKM for solving effectively $3D$ mechanical boundary-value problems [7–9].

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